Year 2023-2024

The Chinese University of Hong Kong

Department of Mathematics

MMAT 5340 Probability and Stochastic Analysis

**Homework 2: Conditional Expectations**

Due Date: 23:59 pm on Tuesday, January 30th, 2024.

Please submit your homework on Blackboard

1. Let $X$ and $Y$ be two random variables with the same discrete uniform distributions i.e. their probability mass functions are given by

$$\mathbb{P}(X = k) = \mathbb{P}(Y = k) = \frac{1}{N}, \quad k \in \{1, 2, \ldots, N\}.$$

Suppose that $X$ and $Y$ are independent. Compute $\mathbb{E}[X + Y \mid X]$.

**Hint:** you may want to recall the elementary formula

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

2. Suppose that $X$ and $Y$ are independent random variables, both with the standard normal distribution i.e. $X, Y \sim \mathcal{N}(0, 1)$. For $\rho \in [-1, 1]$, define

$$Z := \sqrt{1-\rho^2} X + \rho Y.$$

Show that $\mathbb{E}[Z \mid Y] = \rho Y$.

3. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X$ and $Y$ be two independent continuous random variables with joint density function $\rho_{X,Y}(x, y)$ and marginal density functions $\rho_X(x)$, $\rho_Y(y)$.

Let $g(X, Y)$ be a function with $\mathbb{E}[\ |g(X, Y)|\ ] < \infty$ and we define

$$f(y) := \mathbb{E}[g(X, y)].$$

Show that

$$\mathbb{E}[g(X, Y) \mid Y] = f(Y).$$

**Hint:** Use Fubini's theorem; also, the proof of Example 1.16 (ii) in the lecture notes may be helpful.

**Hint:** Recall that for a random variable $Z$ that is $\sigma(Y)$-measurable, there exists some measurable function $h$ such that $Z = h(Y)$. You may use this result without proof.

**Remark.** *This problem has a more general version: Let $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$. Assume that $X$ is independent from $\mathcal{G}$ and that $Y$ is $\mathcal{G}$-measurable. Can you show that $\mathbb{E}[g(X,Y) \mid Y] = f(Y)$?*

*(This will not be counted for marks; just a fun digression.)*